On the metric dimension and fractional metric dimension for hierarchical product of graphs

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Abstract

A set of vertices W *resolves* a graph G if every vertex of G is uniquely determined by its vector of distances to the vertices in W. The *metric dimension* for G, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of G. In order to study the metric dimension for the hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ of two rooted graphs $G_2^{u_2}$ and $G_1^{u_1}$, we first introduce a new parameter, the *rooted metric dimension* $\operatorname{rdim}(G_1^{u_1})$ for a rooted graph $G_1^{u_1}$. If G_1 is not a path with an end-vertex u_1, we show that $\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1})$, where $|V(G_2)|$ is the order of G_2. If G_1 is a path with an end-vertex u_1, we obtain some tight inequalities for $\dim(G_2^{u_2} \sqcap G_1^{u_1})$. Finally, we show that similar results hold for the fractional metric dimension.

Key words: resolving set; metric dimension; resolving function; fractional metric dimension; hierarchical product; binomial tree.

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1 Introduction

All graphs considered in this paper are nontrivial and connected. For a graph G, we often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. For any two vertices u and v of G, denote by $d_G(u, v)$ the distance between u and v in G, and write $R_G\{u, v\} = \{w \mid w \in V(G), d_G(u, w) \neq d_G(v, w)\}$. If the graph G is clear from the context, the notations $d_G(u, v)$ and $R_G\{u, v\}$ will be written $d(u, v)$ and $R\{u, v\}$, respectively. A subset W of $V(G)$ is a *resolving set* of G if $W \cap R\{u, v\} \neq \emptyset$ for any two distinct vertices u and v. A *metric basis* of G is a resolving set of G with minimum cardinality. The cardinality of a metric basis of G is the *metric dimension* for G, denoted by $\dim(G)$.

Metric dimension was introduced independently by Harary and Melter [15], and by Slater [24]. As a graph parameter it has numerous applications, among them are computer science and robotics [18], network discovery and verification [5], strategies

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for the Mastermind game [8] and combinatorial optimization [23]. Metric dimension has been heavily studied, see [3] for a number of references on this topic.

The problem of finding the metric dimension for a graph was formulated as an integer programming problem independently by Chartrand et al. [7], and by Currie and Oellermann [10]. In graph theory, fractionalization of integer-valued graph theoretic concepts is an interesting area of research (see [22]). Currie and Oellermann [10] and Fehr et al. [11] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem. Arumugam and Mathew [1] initiated the study of the fractional metric dimension for graphs. For more information, see [2, 12, 13].

Let $g : V(G) \longrightarrow [0, 1]$ be a real value function. For $W \subseteq V(G)$, denote $g(W) = \sum_{v\in W} g(v)$. The *weight* of g is defined by $|g| = g(V(G))$. We call g a *resolving function* of G if $g(R\{u, v\}) \geq 1$ for any two distinct vertices u and v. The minimum weight of a resolving function of G is called the *fractional metric dimension* for G, denoted by $\dim_f(G)$.

It was noted in [14, p. 204] and [18] that determining the metric dimension for a graph is an NP-complete problem. So it is desirable to reduce the computation for the metric dimension for product graphs to the computation for some parameters of the factor graphs; see [6] for cartesian products, [16] for lexicographic products, and [25] for corona products. Recently, the fractional metric dimension for the above three products was studied in [2, 12, 13].

In order to model some real-life complex networks, Barrière et al. [4] introduced the hierarchical product of graphs and showed that it is associative. A *rooted graph* G^u is the graph G in which one vertex u, called *root vertex*, is labeled in a special way to distinguish it from other vertices. Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. The *hierarchical product* $G_2^{u_2} \sqcap G_1^{u_1}$ is the rooted graph with the vertex set $\{x_2x_1 \mid x_i \in V(G_i), i = 1, 2\}$, having the root vertex u_2u_1, where x_2x_1 is adjacent to y_2y_1 whenever $x_2 = y_2$ and $\{x_1, y_1\} \in E(G_1)$, or $x_1 = y_1 = u_1$ and $\{x_2, y_2\} \in E(G_2)$. See [17, 19, 20, 21] for more information.

In this paper, we study the (fractional) metric dimension for the hierarchical product $G_2^{u_2} \sqcap G_1^{u_1}$ of rooted graphs $G_2^{u_2}$ and $G_1^{u_1}$. In Section 2, we introduce a new parameter, the rooted metric dimension $\mathrm{rdim}(G^u)$ for a rooted graph G^u. If G_1 is not a path with an end-vertex u_1, we show that $\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \mathrm{rdim}(G_1^{u_1})$. If G_1 is a path with an end-vertex u_1, we obtain some tight inequalities for $\dim(G_2^{u_2} \sqcap G_1^{u_1})$. In Section 3, we show that similar results hold for the fractional metric dimension.

2 Metric dimension

In order to study the metric dimension for the hierarchical product of graphs, we first introduce the rooted metric dimension for a rooted graph.

A *rooted resolving set* of a rooted graph G^u is a subset W of $V(G)$ such that $W \cup \{u\}$ is a resolving set of G. A *rooted metric basis* of G^u is a rooted resolving set of G^u with the minimum cardinality. The cardinality of a rooted metric basis of G^u is called *rooted metric dimension* for G^u, denoted by $\mathrm{rdim}(G^u)$. The following

observation is obvious.

Observation 2.1 *If there exists a metric basis of G containing u, then* $\mathrm{rdim}(G^u) = \dim(G) - 1$. *If any metric basis of G does not contain u, then* $\mathrm{rdim}(G^u) = \dim(G)$.

For graphs H_1 and H_2 we use $H_1 \cup H_2$ to denote the disjoint union of H_1 and H_2 and $H_1 + H_2$ to denote the graph obtained from the disjoint union of H_1 and H_2 by joining every vertex of H_1 with every vertex of H_2.

Observation 2.2 *Let G be a graph of order n. Then $1 \leq \dim(G) \leq n-1$. Moreover,*
(i) $\dim(G) = 1$ *if and only if G is the path P_n of length n.*
(ii) $\dim(G) = n - 1$ *if and only if G is the complete graph K_n on n vertices.*

Proposition 2.3 [7, Theorem 4] *Let G be a graph of order $n \geq 4$. Then* $\dim(G) = n - 2$ *if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K_t}$ $(s \geq 1, t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$, where $\overline{K_t}$ is a null graph and $K_{s,t}$ is a complete bipartite graph.*

Proposition 2.4 *Let G^u be a rooted graph of order n. Then* $0 \leq \mathrm{rdim}(G^u) \leq n-2$. *Moreover,*
(i) $\mathrm{rdim}(G^u) = 0$ *if and only if $G = P_n$ and u is one of its end-vertices.*
(ii) $\mathrm{rdim}(G^u) = n - 2$ *if and only if $G = K_n$, or $G = K_{1,n-1}$ and u is the centre.*

Proof. If G is a complete graph, by Observation 2.2 (ii) we have $\dim(G) = n - 1$. Observation 2.1 implies that $\mathrm{rdim}(G^u) = n - 2$. If G is not a complete graph, then $1 \leq \dim(G) \leq n - 2$, which implies that $0 \leq \mathrm{rdim}(G^u) \leq n - 2$ by Observation 2.1.

(i) Since $\mathrm{rdim}(G^u) = 0$ if and only if $\{u\}$ is a metric basis of G, by Observation 2.2 (i), (i) holds.

(ii) Suppose $\mathrm{rdim}(G^u) = n-2$. Then $\dim(G) = n-1$ or $n-2$. If $\dim(G) = n-1$, then $G = K_n$. Now we consider $\dim(G) = n - 2$. If $n = 3$, then $\dim(G) = 1$, which implies that $G = K_{1,2}$ and u is the centre. Now suppose $n \geq 4$. Then G is one of graphs listed in Proposition 2.3. If $s, t \geq 2$ or $G = K_s + (K_1 \cup K_t)$, then there exists a metric basis containing u, which implies that $\mathrm{rdim}(G^u) = n - 3$, a contradiction. Hence $G = K_{1,n-1}$. Since any metric basis of $K_{1,n-1}$ does not contain the centre, the vertex u is the centre of $K_{1,n-1}$. The converse is routine. $\square$

Next, we study the metric dimension for the hierarchical product of graphs.

Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. For any two vertices x_2x_1 and y_2y_1 of $G_2^{u_2} \sqcap G_1^{u_1}$, observe that

$$d(x_2x_1, y_2y_1) = \begin{cases} d_{G_1}(x_1, y_1), & \text{if } x_2 = y_2, \\ d_{G_2}(x_2, y_2) + d_{G_1}(x_1, u_1) + d_{G_1}(y_1, u_1), & \text{if } x_2 \neq y_2. \end{cases} \tag{1}$$

Lemma 2.5 *Let x_2x_1 and y_2y_1 be two distinct vertices of $G_2^{u_2} \sqcap G_1^{u_1}$.*
(i) *If $x_2 = y_2$, then*

$$R\{x_2x_1, y_2y_1\} = \begin{cases} \{x_2z \mid z \in R_{G_1}\{x_1, y_1\}\}, & \text{if } u_1 \notin R_{G_1}\{x_1, y_1\}, \\ V(G_2^{u_2} \sqcap G_1^{u_1}) \setminus \{x_2z \mid z \notin R_{G_1}\{x_1, y_1\}\}, & \text{if } u_1 \in R_{G_1}\{x_1, y_1\}. \end{cases}$$

(ii) *If $x_2 \neq y_2$, then $\{x_2z, y_2z\} \cap R\{x_2x_1, y_2y_1\} \neq \emptyset$ for any $z \in V(G_1)$.*

Proof. (i) If $u_1 \notin R_{G_1}\{x_1, y_1\}$, then $d_{G_1}(x_1, u_1) = d_{G_1}(y_1, u_1)$. By (1), the inequality $d(x_2x_1, z_2z_1) \neq d(y_2y_1, z_2z_1)$ holds if and only if $z_2 = x_2$ and $d_{G_1}(x_1, z_1) \neq d_{G_1}(y_1, z_1)$. It follows that $R\{x_2x_1, y_2y_1\} = \{x_2z \mid z \in R_{G_1}\{x_1, y_1\}\}$. If $u_1 \in R_{G_1}\{x_1, y_1\}$, then $d_{G_1}(x_1, u_1) \neq d_{G_1}(y_1, u_1)$. By (1), the equality $d(x_2x_1, z_2z_1) = d(y_2y_1, z_2z_1)$ holds if and only if $z_2 = x_2$ and $d_{G_1}(x_1, z_1) = d_{G_1}(y_1, z_1)$. It follows that $R\{x_2x_1, y_2y_1\} = V(G_2^{u_2} \sqcap G_1^{u_1}) \setminus \{x_2z \mid z \notin R_{G_1}\{x_1, y_1\}\}$.

(ii) Suppose $x_2z \notin R\{x_2x_1, y_2y_1\}$. Then $d(x_2x_1, x_2z) = d(y_2y_1, x_2z)$. By (1),

$$d_{G_1}(x_1, z) = d_{G_2}(y_2, x_2) + d_{G_1}(y_1, u_1) + d_{G_1}(z, u_1) \geq d_{G_2}(x_2, y_2) + d_{G_1}(y_1, z),$$

which implies that

$$d_{G_2}(x_2, y_2) + d_{G_1}(x_1, u_1) + d_{G_1}(z, u_1) \geq 2d_{G_2}(x_2, y_2) + d_{G_1}(y_1, z) > d(y_2y_1, y_2z).$$

Hence, $y_2z \in R\{x_2x_1, y_2y_1\}$, as desired. □

Lemma 2.6 *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. Then*

$$\operatorname{rdim}(G_2^{u_2} \sqcap G_1^{u_1}) \geq |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1}).$$

Proof. Let $\overline{W}$ be a rooted metric basis of $G_2^{u_2} \sqcap G_1^{u_1}$. For $v \in V(G_2)$, write $\overline{W}_v = \{z \mid vz \in \overline{W}\}$. For any two distinct vertices x, y of G_1, there exists a vertex wz in $\overline{W} \cup \{u_2u_1\}$ such that $d(vx, wz) \neq d(vy, wz)$. If $w = v$, by (1) we get $d_{G_1}(x, z) \neq d_{G_1}(y, z)$, which implies that $z \in (\overline{W}_v \cup \{u_1\}) \cap R_{G_1}\{x, y\}$. If $w \neq v$, by (1) we have $d_{G_1}(x, u_1) \neq d_{G_1}(y, u_1)$, which implies that $u_1 \in R_{G_1}\{x, y\}$. Therefore, we have $(\overline{W}_v \cup \{u_1\}) \cap R_{G_1}\{x, y\} \neq \emptyset$, which implies that $\overline{W}_v$ is a rooted resolving set of $G_1^{u_1}$. Hence

$$\operatorname{rdim}(G_2^{u_2} \sqcap G_1^{u_1}) = |\overline{W}| = \sum_{v \in V(G_2)} |\overline{W}_v| \geq |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1}),$$

as desired. □

Theorem 2.7 *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If G_1 is not a path with an end-vertex u_1, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1}).$$

Proof. By Lemma 2.6, we only need to prove that

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) \leq |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1}). \tag{2}$$

Let W be a rooted metric basis of $G_1^{u_1}$. Then $W \neq \emptyset$. Write $\overline{W} = \{vw \mid v \in V(G_2), w \in W\}$. Note that $|\overline{W}| = |V(G_2)| \cdot \operatorname{rdim}(G_1^{u_1})$. In order to prove (2), we only need to show that $\overline{W}$ is a resolving set of $G_2^{u_2} \sqcap G_1^{u_1}$. It suffices to show that, for any two distinct vertices x_2x_1 and y_2y_1 of $G_2^{u_2} \sqcap G_1^{u_1}$,

$$\overline{W} \cap R\{x_2x_1, y_2y_1\} \neq \emptyset. \tag{3}$$

If $x_2 = y_2$ and $u_1 \notin R_{G_1}\{x_1, y_1\}$, then $W \cap R_{G_1}\{x_1, y_1\} \neq \emptyset$, by Lemma 2.5 (i) we obtain (3). If $x_2 = y_2$ and $u_1 \in R_{G_1}\{x_1, y_1\}$, by (1) we have $vw \in \overline{W} \cap R\{x_2x_1, y_2y_1\}$ for any $v \neq x_2$ and any $w \in W$, which implies that (3) holds. If $x_2 \neq y_2$, then (3) holds by Lemma 2.5 (ii). We accomplish our proof. $\square$

Combining Observation 2.1 and Theorem 2.7, we have the following result.

Corollary 2.8 *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs.*

(i) *If there exists a metric basis of G_1 containing u_1 and G_1 is not a path, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)|(\dim(G_1) - 1).$$

(ii) *If any metric basis of G_1 does not contain u_1, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \dim(G_1).$$

The *binomial tree* T_n is the hierarchical product of n copies of the complete graph on two vertices, which is a useful data structure in the context of algorithm analysis and designs [9]. It was proved that the metric dimension for a tree can be expressed in terms of its parameters in [7, 15, 24].

Corollary 2.9 *Let $n \geq 2$. Then $\dim(T_n) = 2^{n-2}$.*

Proof. Note that $\dim(T_2) = 1$. Now suppose $n \geq 3$. Since $T_n = (K_2^0 \sqcap \cdots \sqcap K_2^0) \sqcap (K_2^0 \sqcap K_2^0)$ and $\mathrm{rdim}(K_2^0 \sqcap K_2^0) = 1$, the desired result follows by Theorem 2.7. $\square$

We always assume that 0 is one end-vertex of P_n. In the remaining of this section, we shall prove some tight inequalities for $\dim(G^u \sqcap P_n^0)$.

Proposition 2.10 *Let G^u be a rooted graph with diameter d. Then*

$$\dim(G^u \sqcap P_n^0) \leq \dim(G^u \sqcap P_{n+1}^0) \text{ for } 1 \leq n \leq d-1, \tag{4}$$
$$\dim(G^u \sqcap P_n^0) = \dim(G^u \sqcap P_{n+1}^0) \text{ for } n \geq d. \tag{5}$$

Proof. If $G = K_2$, then $G^u \sqcap P_n^0$ is the path, which implies that (5) holds. Now we only consider $|V(G)| \geq 3$. Suppose that $\overline{W}_n$ is a metric basis of $G^u \sqcap P_n^0$. Let $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$. Define $\pi_n : V(G^u \sqcap P_{n+1}^0) \longrightarrow V(G^u \sqcap P_n^0)$ by

$$\pi_n(vz_i) = \begin{cases} vz_{n-1}, & \text{if } i = n, \\ vz_i, & \text{if } i \leq n-1. \end{cases}$$

Then $\pi_n(\overline{W}_{n+1})$ is a resolving set of $G^u \sqcap P_n^0$, which implies that $\dim(G^u \sqcap P_n^0) \leq \dim(G^u \sqcap P_{n+1}^0)$ for any positive integer n. So (4) holds.

In order to prove (5), we only need to show that $\overline{W}_n$ is a resolving set of $G^u \sqcap P_{n+1}^0$ for $n \geq d$. Pick any two distinct vertices v_1z_i and v_2z_j of $G^u \sqcap P_{n+1}^0$. It suffices to prove that

$$\overline{W}_n \cap R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_2z_j\} \neq \emptyset. \tag{6}$$

We claim that there exist two distinct vertices w_1 and w_2 of G such that $\overline{W}_n \cap \{w_sz_k \mid 0 \leq k \leq n-1\} \neq \emptyset$ for $s \in \{1, 2\}$. Suppose for the contradiction that there

exists a vertex $w \in V(G)$ such that $\overline{W}_n \subseteq \{wz_k \mid 0 \leq k \leq n-1\}$. If the degree of w in G is one, then there exists a path (w, x, y) in G. For any $wz_k \in \overline{W}_n$, we have $d(xz_1, wz_k) = k+2 = d(yz_0, wz_k)$, contrary to the fact that $\overline{W}_n$ is a metric basis of $G^u \sqcap P_n^0$. If the degree of w in G is at least two, pick two distinct neighbors x and y of w in G. Then $d(xz_0, wz_k) = k+1 = d(yz_0, wz_k)$ for any $wz_k \in \overline{W}_n$, a contradiction. Hence our claim is valid.

Now we prove (6). Without loss of generality, we may assume that $0 \leq i \leq j \leq n$. If $j \leq n-1$, then $R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_2z_j\} \supseteq R_{G^u \sqcap P_n^0}\{v_1z_i, v_2z_j\}$; and so (6) holds. Now suppose $j = n$.

Case 1. $v_1 = v_2$. By the claim, the set $\{w_1z_k \mid 0 \leq k \leq n-1\}$ or $\{w_2z_k \mid 0 \leq k \leq n-1\}$ is a subset of $R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_1z_n\}$. So (6) holds.

Case 2. $v_1 \neq v_2$.

Case 2.1. $i = 0$. By the claim, we can choose $w_sz_k \in \overline{W}_n$ with $w_s \neq v_2$. Then

$$d(v_1z_0, w_sz_k) = d_G(v_1, w_s) + k \leq d + k \leq n + k < d_G(v_2, w_s) + n + k = d(v_2z_n, w_sz_k),$$

which implies that $w_sz_k \in R_{G^u \sqcap P_{n+1}^0}\{v_1z_0, v_2z_n\}$. So (6) holds.

Case 2.2. $i \geq 1$. Note that

$$R_{G^u \sqcap P_{n+1}^0}\{v_1z_i, v_2z_n\} = R_{G^u \sqcap P_{n+1}^0}\{v_1z_{i-1}, v_2z_{n-1}\} \supseteq R_{G^u \sqcap P_n^0}\{v_1z_{i-1}, v_2z_{n-1}\}.$$

Then (6) holds. $\square$

Proposition 2.11 *For any rooted graph G^u, we have*

$$\dim(G) \leq \dim(G^u \sqcap P_n^0) \leq |V(G)| - 1. \tag{7}$$

Proof. Let z be the other end-vertex of P_n. Fix a vertex $v_0 \in V(G)$ and write $\overline{S} = \{vz \mid v \in V(G) \setminus \{v_0\}\}$. Since $\{z\}$ is a resolving set of P_n, the set $\overline{S}$ resolves $G \sqcap P_n$ by (1). Hence $\dim(G^u \sqcap P_n^0) \leq |\overline{S}| = |V(G)| - 1$. Since G^u is isomorphic to $G^u \sqcap P_1^0$, Proposition 2.10 implies that $\dim(G) \leq \dim(G^u \sqcap P_n^0)$. $\square$

For $m \geq 2$, we have $\dim(K_m^u \sqcap P_n^0) = m - 1$. This shows that the inequalities (4) and (7) are tight.

Example 2.12 *For $m, n \geq 2$, we have* $\dim(P_m^u \sqcap P_n^0) = 2$.

Proof. Write $P_k = (z_0 = 0, z_1, \ldots, z_{k-1})$. Then $\{z_0z_{n-1}, z_{m-1}z_{n-1}\}$ is a resolving set of $P_m^u \sqcap P_n^0$. $\square$

Example 2.13 *Let C_m be the cycle with length m. Then* $\dim(C_m^u \sqcap P_n^0) = 2$.

Proof. Let $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$ and $C_m = (c_0, c_1, \ldots, c_{m-1}, c_0)$. Then $\{c_0z_{n-1}, c_1z_{n-1}\}$ is a resolving set of $C_m^u \sqcap P_n^0$. $\square$

3 Fractional metric dimension

In order to study the fractional metric dimension for the hierarchical product of graphs, we first introduce the fractional rooted metric dimension for a rooted graph.

Similar to the fractionalization of metric dimension, we give a fractional version of the rooted metric dimension for a rooted graph. Let G^u be a rooted graph of order n. Write

$$\mathcal{P}^u = \{\{v, w\} \mid v, w \in V(G), v \neq w, d(v, u) = d(w, u)\}.$$

Suppose $\mathcal{P}^u \neq \emptyset$. Write $V(G) \setminus \{u\} = \{v_1, \ldots, v_{n-1}\}$ and $\mathcal{P}^u = \{\alpha_1, \ldots, \alpha_m\}$. Let A^u be the $m \times (n-1)$ matrix with

$$(A^u)_{ij} = \begin{cases} 1, & \text{if } v_j \text{ resolves } \alpha_i, \\ 0, & \text{otherwise.} \end{cases}$$

The integer programming formulation of the rooted metric dimension for G^u is given by

$$\begin{aligned} &\text{Minimize } f(x_1, \ldots, x_{n-1}) = x_1 + \cdots + x_{n-1} \\ &\text{Subject to } A^u\mathbf{x} \geq \mathbf{1} \end{aligned}$$

where $\mathbf{x} = (x_1, \ldots, x_{n-1})^{\mathrm{T}}, x_i \in \{0, 1\}$ and $\mathbf{1}$ is the $m \times 1$ column vector all of whose entries are 1. The optimal solution of the linear programming relaxation of the above integer programming problem, where we replace $x_i \in \{0, 1\}$ by $x_i \in [0, 1]$, gives the *fractional rooted metric dimension* for G^u, which we denote by $\mathrm{rdim}_f(G^u)$.

Let G^u be a rooted graph which is not a path with an end-vertex u. A *rooted resolving function* of a rooted graph G^u is a real value function $g : V(G) \longrightarrow [0, 1]$ such that $g(R\{v, w\}) \geq 1$ for each $\{v, w\} \in \mathcal{P}^u$. The *fractional rooted metric dimension* for G^u is the minimum weight of a rooted resolving function of G^u.

Proposition 3.1 *Let G^u be a rooted graph which is not a path with an end-vertex u. Then*
(i) $\mathrm{rdim}_f(G^u) \leq \mathrm{rdim}(G^u)$.
(ii) $\mathrm{rdim}_f(G^u) \leq \frac{|V(G)|-1}{2}$.
(iii) $\dim_f(G) - 1 \leq \mathrm{rdim}_f(G^u) \leq \dim_f(G)$.

Proof. (i) Let W be a rooted metric basis of G^u. Define $g : V(G) \longrightarrow [0, 1]$ by

$$g(v) = \begin{cases} 1, & \text{if } v \in W, \\ 0, & \text{if } v \notin W. \end{cases}$$

For any $\{x, y\} \in \mathcal{P}^u$, there exists a vertex $v \in W$ such that $d(x, v) \neq d(y, v)$. Then $g(R\{x, y\}) \geq g(v) = 1$, which implies that g is a rooted resolving function of G^u. Hence $\mathrm{rdim}_f(G^u) \leq |g| = |W| = \mathrm{rdim}(G^u)$.

(ii) The function $g : V(G) \longrightarrow [0, 1]$ defined by

$$g(v) = \begin{cases} 0, & \text{if } v = u, \\ \frac{1}{2}, & \text{if } v \neq u \end{cases}$$

is a rooted resolving function of G^u. Hence $\text{rdim}_f(G^u) \leq \frac{|V(G)|-1}{2}$.

(iii) It is clear that $\text{rdim}_f(G^u) \leq \dim_f(G)$. Let g be a rooted resolving function of G^u. Then the function $h : V(G) \longrightarrow [0, 1]$ defined by

$$h(v) = \begin{cases} 1, & \text{if } v = u, \\ g(v), & \text{if } v \neq u \end{cases}$$

is a resolving function of G. Hence $\dim_f(G) \leq \text{rdim}_f(G^u) + 1$, as desired. □

If u is not an end-vertex of the path P_n, then $\text{rdim}_f(P_n^u) = \text{rdim}(P_n^u) = \dim_f(P_n) = 1$, which implies that the upper bounds in Proposition 3.1 (i) and (iii) are tight. The fact that $\text{rdim}_f(K_n^u) = \frac{n-1}{2}$ shows that the inequality in Proposition 3.1 (ii) is tight.

Next, we study the fractional metric dimension for the hierarchical product of graphs.

For two rooted graphs $G_1^{u_1}$ and $G_2^{u_2}$, write

$$\mathcal{P}^{u_1} = \{\{x, y\} \subseteq V(G_1) \mid x \neq y, d_{G_1}(x, u_1) = d_{G_1}(y, u_1)\},$$
$$\overline{\mathcal{P}}^{u_2u_1} = \{\{x_2x_1, y_2y_1\} \subseteq V(G_2^{u_2} \sqcap G_1^{u_1}) \mid x_2x_1 \neq y_2y_1, d(x_2x_1, u_2u_1) = d(y_2y_1, u_2u_1)\}.$$

Lemma 3.2 *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If G_1 is not a path with an end-vertex u_1, then*

$$\text{rdim}_f(G_2^{u_2} \sqcap G_1^{u_1}) \geq |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}).$$

Proof. Suppose that $\overline{g}$ is a rooted resolving function of $G_2^{u_2} \sqcap G_1^{u_1}$ with weight $\text{rdim}_f(G_2^{u_2} \sqcap G_1^{u_1})$. For each $z \in V(G_2)$, define

$$\overline{g}_z : V(G_1) \longrightarrow [0, 1], \quad x \longmapsto \overline{g}(zx).$$

Write $\overline{\mathcal{P}}^{u_1} = \{\{zx, zy\} \mid z \in V(G_2), \{x, y\} \in \mathcal{P}^{u_1}\}$. By (1), we have $\overline{\mathcal{P}}^{u_1} \subseteq \overline{\mathcal{P}}^{u_2u_1}$. Hence $\overline{g}_z(R_{G_1}\{x, y\}) \geq 1$ for any $\{x, y\} \in \mathcal{P}^{u_1}$, which implies that $|\overline{g}_z| \geq \text{rdim}_f(G_1^{u_1})$. Consequently,

$$\text{rdim}_f(G_2^{u_2} \sqcap G_1^{u_1}) = |\overline{g}| = \sum_{z \in V(G_2)} |\overline{g}_z| \geq |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}),$$

as desired. □

Theorem 3.3 *Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If G_1 is not a path with an end-vertex u_1, then*

$$\dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}).$$

Proof. Combining Proposition 3.1 and Lemma 3.2, we only need to prove that

$$\dim_f(G_2^{u_2} \sqcap G_1^{u_1}) \leq |V(G_2)| \cdot \text{rdim}_f(G_1^{u_1}). \tag{8}$$

By Proposition 2.4 we have $\mathcal{P}^{u_1} \neq \emptyset$. Let g be a rooted resolving function of G_1 with weight $\text{rdim}_f(G_1^{u_1})$. Define

$$\overline{g} : V(G_2^{u_2} \sqcap G_1^{u_1}) \longrightarrow [0, 1], \quad x_2x_1 \longmapsto g(x_1).$$

We shall show that, for any two distinct vertices x_2x_1 and y_2y_1 of $G_2^{u_2} \sqcap G_1^{u_1}$,

$$\overline{g}(R\{x_2x_1, y_2y_1\}) \geq 1. \tag{9}$$

Case 1. $x_2 = y_2$. If $u_1 \notin R_{G_1}\{x_1, y_1\}$, by Lemma 2.5 we get $R\{x_2x_1, y_2y_1\} = \{x_2z \mid z \in R_{G_1}\{x_1, y_1\}\}$, which implies that $\overline{g}(R\{x_2x_1, y_2y_1\}) = g(R_{G_1}\{x_1, y_1\})$. Since $\{x_1, y_1\} \in \mathcal{P}^{u_1}$, we obtain (9). If $u_1 \in R_{G_1}\{x_1, y_1\}$, by Lemma 2.5 we have $R\{x_2x_1, y_2y_1\} \supseteq \{vz \mid z \in V(G_1)\}$ for any $v \in V(G_2) \setminus \{x_2\}$, which implies that $\overline{g}(R\{x_2x_1, y_2y_1\}) \geq |g|$, so (9) holds.

Case 2. $x_2 \neq y_2$. Write $W = \{z \mid x_2z \in R\{x_2x_1, y_2y_1\}\}$ and $S = \{z \mid y_2z \in R\{x_2x_1, y_2y_1\}\}$. By Lemma 2.5 we have $W \cup S = V(G_1)$. Then

$$\overline{g}(R\{x_2x_1, y_2y_1\}) \geq \sum_{z \in W} \overline{g}(x_2z) + \sum_{z \in S} = g(W) + g(S)\overline{g}(y_2z) \geq |g|,$$

which implies that (9) holds.

Therefore, $\overline{g}$ is a resolving function of $G_2^{u_2} \sqcap G_1^{u_1}$, which implies that $\dim_f(G_2^{u_2} \sqcap G_1^{u_1}) \leq |\overline{g}|$. Since $|\overline{g}| = |V(G_2)| \cdot \mathrm{rdim}_f(G_1^{u_1})$, we obtain (8). Our proof is accomplished. $\square$

Corollary 3.4 *Let $n \geq 2$. Then $\dim_f(T_n) = 2^{n-2}$.*

Proof. It is immediate from Theorem 3.3. $\square$

By Corollaries 2.9 and 3.4, the binomial tree T_n is a graph whose metric dimension is equal to its fractional metric dimension.

Finally, we shall prove some tight inequalities for $\dim_f(G^u \sqcap P_n^0)$.

Proposition 3.5 *For any rooted graph G^u, we have*

$$\dim_f(G) \leq \dim_f(G^u \sqcap P_n^0) \leq \dim_f(G^u \sqcap P_{n+1}^0) \leq \frac{|V(G)|}{2}.$$

Proof. Write $P_n = (z_0 = 0, z_1, \ldots, z_{n-1})$. For a resolving function $\overline{g}_{n+1}$ of $G^u \sqcap P_{n+1}^0$, we define $\overline{g}'_{n+1} : V(G^u \sqcap P_n^0) \longrightarrow [0, 1]$ by

$$\overline{g}'_{n+1}(x_2x_1) = \begin{cases} \overline{g}_{n+1}(x_2z_{n-1}) + \overline{g}_{n+1}(x_2z_n), & \text{if } x_1 = z_{n-1}, \\ \overline{g}_{n+1}(x_2x_1), & \text{if } x_1 \neq z_{n-1}. \end{cases}$$

Then $\overline{g}'_{n+1}$ is a resolving function of $G^u \sqcap P_n^0$. Since $|\overline{g}'_{n+1}| = |\overline{g}_{n+1}|$, we have

$$\dim_f(G) = \dim_f(G^u \sqcap P_1^0) \leq \dim_f(G^u \sqcap P_n^0) \leq \dim_f(G^u \sqcap P_{n+1}^0).$$

Define $\overline{h} : V(G^u \sqcap P_{n+1}^0) \longrightarrow [0, 1]$ by

$$\overline{h}(x_2x_1) = \begin{cases} \frac{1}{2}, & \text{if } x_1 = z_n, \\ 0, & \text{if } x_1 \neq z_n. \end{cases}$$

Then $\overline{h}$ is a resolving function of $G^u \sqcap P_{n+1}^0$ with weight $\frac{|V(G)|}{2}$. Hence $\dim_f(G^u \sqcap P_{n+1}^0) \leq \frac{|V(G)|}{2}$. $\square$

For $m \geq 2$, we have $\dim_f(K_m^u \sqcap P_n^0) = \frac{m}{2}$. This shows that all the inequalities in Proposition 3.5 are tight.

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